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A J-INTEGRAL METHOD FOR CALCULATING STEADY-STATE MATRIX CRACKING STRESSES IN COMPOSITES

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A general expression for the steady-state matrix cracking stress in reinforced brittle matrix composites is derived using a J-integral analysis. The result is expressed in terms of the stress–displacement relation that characterizes the stretching of crack bridging ligaments. The influence of residual stress is assessed and a condition for spontaneous matrix cracking due to residual tensile stress in the matrix is evaluated. Results of the analysis are compared with independent solutions for composites containing unbonded reinforcing fibers.

1. Introduction

If the strengths of crack bridging ligaments in a composite material exceed a critical value, a crack can be made to extend indefinitely in the matrix without the ligaments rupturing in its wake. Further loading causes formation of periodic matrix cracks, with separation dictated by a characteristic transfer length associated with the bridging ligaments (Aveston and Kelly, 1973; Aveston et al., 1971). This failure mechanism has been observed in several important composite systems, including reinforced cements, glasses, and glass ceramics (Aveston et al., 1971; Marshall and Evans, 1985; Sambell et al., 1972; Phillips, 1972; Brennan and Prewo, 1982; Prewo and Brennan, 1982; Ali and Grimer, 1969; Majumdar, 1970; DeVekey and Majumdar 1968; Allen, 1971). An analogous mechanism has also been observed in connection with the cracking of thin glass films on metal substrates (Raj, 1987). The stress required to extend the first matrix crack (i.e., the onset of damage) is a decreasing function of the crack size if the crack is small, but approaches a constant "steady-state" value for large cracks (Marshall et al, 1985). In this paper we present a simple derivation of a general solution for the steady-state matrix cracking stress. The solution, which is obtained by use of the J-integral (Rice, 1968), is equivalent to one obtained recently by Rose (1987), who evaluated changes in configurational energy involved in transporting a strip of material from a location far ahead of the crack tip to a position far behind the tip.

The steady-state matrix cracking stress has been calculated previously for some specific, relatively straightforward ligament bridging mechanisms. Aveston, Cooper, and Kelly (1971) first analyzed bridging that involves frictional sliding between fibers and matrix, in the limit of small frictional stress. More recently, Budiansky, Hutchinson, and Evans (1986) obtained a general solution for frictional sliding, valid over the full range of frictional stresses, from the small stress limit of Aveston, Cooper, and Kelly to the large stress limit at which sliding does not occur and the bridging ligaments act as linear springs. These analyses involved somewhat detailed calculation of energy changes, similar to the approach adopted by Rose (1987). An alternative, stress intensity approach has been used to evaluate both steady-state and nonsteady-state cracking for frictional and linear-spring ligaments (Marshall et al., 1985; Marshall and Cox, 1987). However, evaluation of the steady-state matrix cracking stress by this approach entails numerical solution of an integral equation. An appealing feature of the solution discussed in this paper (in addition to its simple

analytic form and generality) is that it is expressed directly in terms of the stress-displacement relation that describes the stretching of the bridging ligaments. This form of the solution allows changes in matrix cracking stress due to changes in the bridging mechanism or to the presence of residual stresses to be readily deduced.

2. General solution for steady-state matrix cracking

The opening of a crack that is bridged by reinforcements involves stretching of ligaments between the crack surfaces. This stretching may be characterized by a relation between the stress, σ_f, in the ligament and an average local crack opening displacement, u, as depicted in Fig. 1. The form of this relation depends on the details of the bridging mechanism and reflects properties such as ligament deformation, reinforcement/matrix debonding and frictional sliding, as well as elastic stretching of the reinforcement. The peak value, $\sigma_f = S$, represents the strength of the ligaments. The decreasing portion of the $\sigma_f(u)$ curve depends on the nature and location of reinforcement failure. This is important for determining the toughness of composites in which reinforcements rupture in the wake of the crack, but it does not influence steady-state matrix cracking.

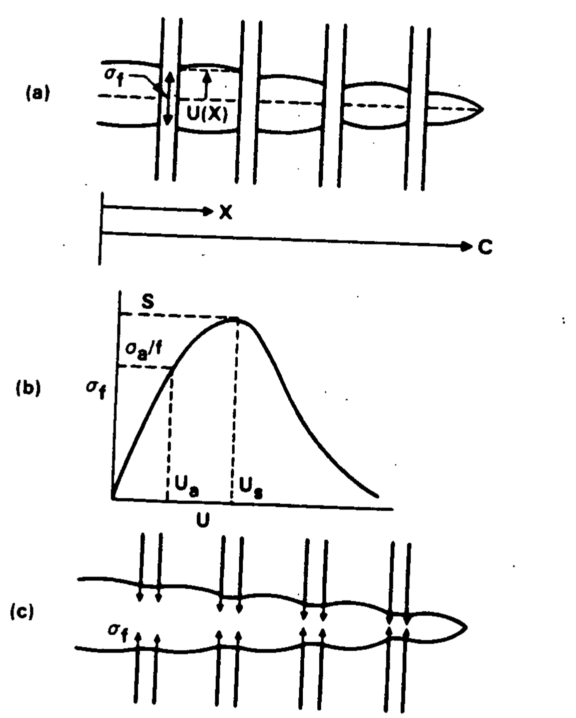


Fig. 1. (a) Crack bridged by reinforcing ligaments; (b) Schematic of stress–displacement relation for stretching of bridging ligaments; (c) Crack with bridging ligaments replaced by surface tractions.

The influence of bridging ligaments on the stress anywhere in the body can be evaluated by replacing the ligaments with crack surface tractions equal in magnitude to the stress, σ_f, in the stretched ligament (Fig. 1(c)). To proceed further, the composite is approximated as a continuum with the average elastic properties of the composite, and with continuous pressure

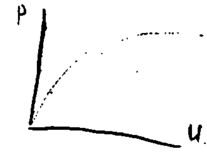
$$p(u) = f\sigma_f(u) \tag{1}$$

acting over the crack surfaces (f is the volume fraction of reinforcements). This continuum approximation requires the crack to be large compared with the dimensions of the microstructural features that give rise to the bridging ligaments. We consider a crack with intact ligaments over its entire surface and subject to remotely applied, uniform, normal, tensile stress, σ_a, that increases monotonically from zero to a value smaller than fS. The crack opening displacement and the crack surface pressure increase monotonically with distance behind the crack tip and, for sufficiently long cracks, approach asymptotic limits equal to u_a and σ_a (p cannot exceed σ_a) at the mouth of the crack (Marshall et al., 1985) (Fig. 2(a)). This is the steady-state matrix crack configuration. The stresses at the crack tip increase as the applied stress increases, but are independent of the total crack length.

For the purpose of calculating the stresses and strains near the crack tip, the remotely applied uniform stress, σ_a, can be replaced by a uniform opening pressure acting over the crack surfaces. Then the resultant crack surface pressure becomes

$$\sigma = \sigma_a - p(u) \tag{2}$$

as depicted in Fig. 2(b). This pressure is maximum at the crack tip and, for the steady-state matrix crack, approaches zero far from the crack tip.



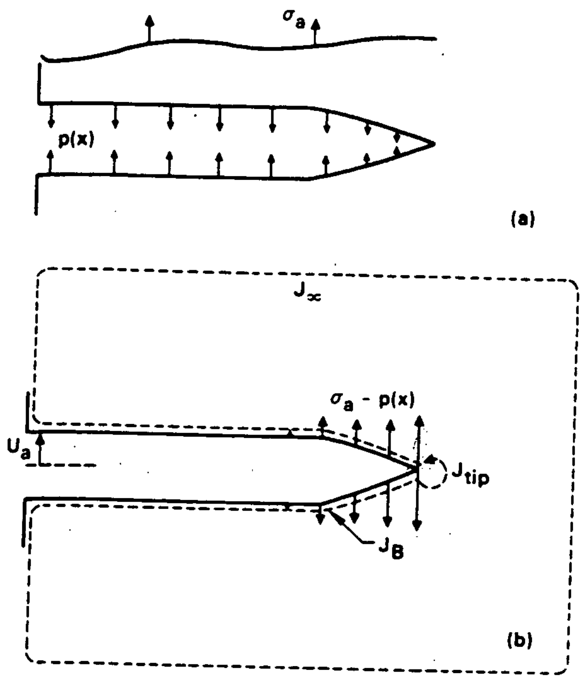


Fig. 2. (a) Steady-state matrix crack loaded with uniform applied stress; (b) Steady-state matrix crack with uniform applied stress replaced by uniform opening pressure acting on crack surfaces.

The crack configuration of Fig. 2(b) is convenient for evaluating the J-integral (Rice, 1968). For the closed path shown in Fig. 2(b), there are three contributions that must sum to zero:

$$J_\infty + J_B + J_{\text{tip}} = 0. \tag{3}$$

The term J_∞ from the path around the stress-free outer boundaries of the body is zero. The contribution from the path over the crack surfaces is (e.g. Lawn and Wilshaw, 1975)

$$J_B = -2\int_0^{u_a} p(u)\,\mathrm{d}u, \tag{4}$$

and the contribution from the path around the crack tip can be expressed

$$J_{\text{tip}} = K^2(1-\nu^2)/E, \tag{5}$$

⇒ 6

where K is the crack tip stress intensity factor and E and ν are the Young's modulus and Poisson's ratio for the composite. Combination of eqns. (2)–(5) yields the result

$$K^2(1-\nu^2)/2E = \sigma_a u_a - \int_0^{u_a} p(u)\,\mathrm{d}u \tag{6}$$

$$= \int_0^{\sigma_a} u\,\mathrm{d}p. \tag{7}$$

Equation (6) is a general expression relating the crack tip stresses (or J_{tip}) to the applied stress. It is noted that this expression is equivalent to the result derived by Rose (1987) and that the right-hand side is the complementary energy density.

The *critical* applied stress, σ_c, for steady-state matrix cracking can be evaluated for any given bridging law by setting $J_{\text{tip}} = J_c$ (or equivalently $K = K_c$) as the criterion for crack growth in the matrix. However, it is necessary to express this fracture criterion in terms of the fracture toughness of the unreinforced matrix. Two approaches have been used for this. In one, the fracture energy is taken to be reduced by the factor $(1-f)$ to reflect the true area of fracture surface created in the matrix (Budiansky, 1986; McCartney, 1987; Kelly and McCartney, 1987), i.e.

$$J_c = J_c^m(1-f) \tag{8}$$

where J_c^m is the fracture energy of the unreinforced matrix. An alternative (upper bound) criterion was obtained by taking the stress intensity factor *in the matrix* to be equal to the critical stress intensity factor, K_c^m, in the unreinforced matrix (Marshall et al., 1985), i.e.

$$K_c = K_c^m E/E_m \tag{9}$$

where E_m is the Young's modulus of the matrix, and the factor E/E_m accounts for the relative magnitudes of the average composite stresses and the actual stress in the matrix. With eqn. (5) this criterion can be expressed

$$J_c = E(K_c^m)^2(1-\nu^2)/E_m^2, \tag{10}$$

which differs from eqn. (8) by the factor $E_m(1-f)/E$ (since $J_c^m = (K_c^m)^2(1-\nu^2)/E_m$). The differences between these two criteria are discussed more fully elsewhere (Marshall et al., 1988). For consistency with other publications, we shall adopt the criterion of eqn. (8) in the following sections.

3. Comparison with alternative solutions: Unbonded fibers

To evaluate the energy changes accompanying steady-state matrix cracking in unbonded fiber reinforced composites, Budiansky, Hutchinson and Evans (1986) calculated the stresses and strains in the bridging fibers and the surrounding matrix using a modified shear lag analysis. By integrating these strains to obtain the crack opening displacement, the stress–displacement relation for the bridging fibers was recently derived in the following form (Marshall et al., 1988):

$$u = \begin{cases} \alpha p & p < p^*,\ u < u^* \\ \beta(p^2 + \alpha^2/4\beta^2) & p > p^*,\ u > u^* \end{cases} \tag{11}$$

where

$$p^* = \alpha/2\beta \tag{12}$$

$$u^* = \alpha^2/2\beta \tag{13}$$

$$\alpha = \frac{RE_m(1-f)}{\rho f E_f E} \tag{14}$$

$$\beta = \frac{RE_m^2(1-f)^2}{4E_f E^2 f^2 \tau} \tag{15}$$

$$\rho^2 = \frac{-4E(1-f)}{E_f(1+\nu_m)[2\log f + (1-f)(3-f)]} \tag{16}$$

with R the fiber radius, E_f the Young's modulus of the fibers, ν_m the Poisson's ratio of the matrix, and τ the sliding frictional stress at the fiber matrix interface. The condition ($u = u^*$, $p = p^*$) defines the onset of sliding between the fibers and matrix. For displacements smaller than u^*, sliding does not occur and the fiber bridges act as linear elastic springs. For $u \gg u^*$, sliding between the fibers and matrix occurs over an area that extends a large distance (compared with R) from the crack surface and the solution corresponds to the large slip limit analyzed in the steady state by Aveston, Cooper and Kelly (1971).

3.1 No-slip limit: Linear springs

If the matrix cracking stress is smaller than p^*, slip does not occur at any of the crack-bridging fibers. The stress–displacement relation is (eqn. 11)

$$u = \alpha p \tag{17}$$

and substitution in eqn. (6) gives the solution for the steady-state matrix cracking stress

$$\sigma_0 = (J_c/\alpha)^{1/2}. \tag{18}$$

With J_c and α defined by eqn. (8) and (14), it is straightforward to show that eqn. (18) is identical to the energy balance solution of Budiansky, Hutchinson and Evans (1986).

3.2 Large slip solution

If the matrix cracking stress is much larger than p^*, the bridging stresses are dominated by the large displacement limit

$$u = \beta p^2. \tag{19}$$

In this case the steady-state matrix cracking stress obtained by substitution in eqn. (6) is

$$\sigma_1 = (3J_c/2\beta)^{1/3}. \tag{20}$$

With J_c and β given by eqns. (8) and (15), this result is identical to the large slip solution of Aveston, Cooper and Kelly (1971).

3.3 General solution

The general solution for steady-state matrix cracking stresses, σ_c, larger than p^* is obtained by substituting the complete expression of eqn. (11) into eqn. (6) yielding

$$J_c = \left(\frac{2\beta}{3}\right)\left[\sigma_c^3 + 3p^*\sigma_c - (p^*)^3\right]. \tag{21}$$

This result can be expressed in terms of the limiting solutions σ_0 and σ_1 for no slip and large slip by making use of the relations

$$\frac{J_c}{2\beta\sigma_0^3} = \frac{p^*}{\sigma_0} = \frac{1}{3}\left(\frac{\sigma_1}{\sigma_0}\right)^3 \tag{22}$$

which follow from eqns. (12), (18) and (20). The resulting expression,

$$\left(\frac{\sigma_1}{\sigma_0}\right)^3 = \left(\frac{\sigma_c}{\sigma_0}\right)^3 + \frac{1}{3}\left(\frac{\sigma_1}{\sigma_0}\right)^6\left(\frac{\sigma_c}{\sigma_0}\right) - \frac{1}{27}\left(\frac{\sigma_1}{\sigma_0}\right)^9, \tag{23}$$

is identical to the energy balance solution of Budiansky, Hutchinson and Evans (1986).

4. The influence of residual stress

Many composites contain residual stresses due to differences in nonelastic strains that occur in the matrix and reinforcements during fabrication (e.g., thermal expansion mismatch, plasticity, phase transformation). The derivation of eqn. (6) is based on a continuum model that requires all dimensions to be large compared with the microstructural features that give rise to the residual stresses and the bridging ligaments. Over such dimensions the average residual stress is zero. Therefore, the derivation of eqn. (6) is not altered in any way by the presence of residual stresses (provided, of course, that further phase transformation or plasticity does not accompany matrix cracking). However, residual stresses do influence σ_c through their effect on the stress-separation function $p(u)$.

The influence of residual stress on the relation $p(u)$ has been evaluated for composites with uniaxially aligned fibers (Marshall and Evans, 1988). In this case the longitudinal residual stresses in the matrix, σ_R^m, and fibers, σ_R^f, are related by

$$\sigma_R^m/E_m = -\sigma_R^f/E_f. \tag{24}$$

One effect of the residual stress is to shift the intercept of the function $p(u)$ along the stress axis by

$$\sigma_R^0 = -\sigma_R^m E/E_m \tag{25}$$

as depicted in Fig. 3. In general, residual stress can also alter the slope of the $p(u)$ relation. Therefore, this relation can be written in the form

$$p(u) = \sigma_R^0 + p_r(u, \sigma_R^0), \tag{26}$$

where the subscript r denotes a modified function in the presence of residual stress, and eqn. (6) becomes

$$K^2(1-\nu^2)/E = 2(\sigma_a - \sigma_R^0)u_a - 2\int_0^{u_a} p_r(u, \sigma_R^0)\,du \tag{27}$$

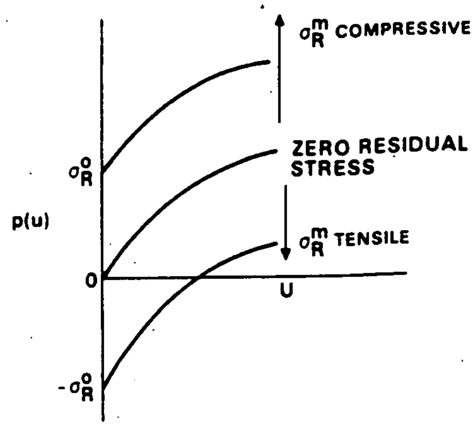


Fig. 3. Influence of residual stress on the stress-displacement relation for ligament stretching.

For certain bridging mechanisms, the effect of residual stress is to translate the function $p(u)$ uniformly along the stress axis. Two examples are fibers that do not experience debonding or sliding and fibers held by friction due to surface roughness at the fiber matrix interface (i.e., the sliding resistance, τ, independent of residual stress). In this case

$$p_r(u, \sigma_R^0) = p_0(u), \tag{28}$$

where $p_0(u)$ is the stress–displacement function in the absence of residual stress, and both u_a and the integral in eqn. (27) are independent of σ_R^0. Therefore, the effect of the residual stress is simply to increase the matrix cracking stress by σ_R^0 (σ_R^0 is positive for *compressive* residual stress in the matrix and negative for tensile residual stress).

Equation (27) with $K = K_c$ provides a general solution for the critical applied stress for matrix cracking in the presence of residual stress. Alternatively, we can solve for the *critical residual stress* that will cause *spontaneous* matrix cracking by setting $\sigma_a = 0$ at $K = K_c$. (In this case σ_R^0 will be negative, i.e., tensile residual matrix stress.)

5. Discussion

An insightful representation of the general relation for steady-state matrix cracking (eqn. (6)) is shown in Fig. 4(a). The right-hand side of eqn. (6)

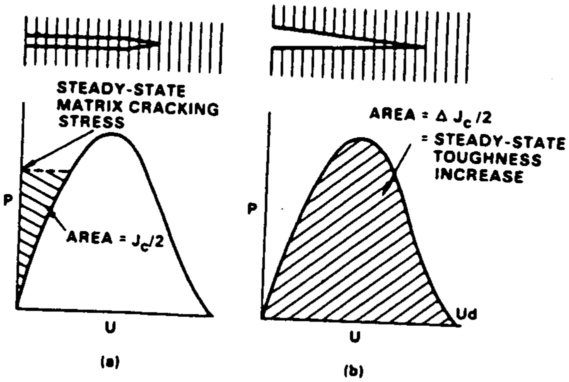


Fig. 4. Representation of (a) steady-state matrix cracking stress, and (b) steady-state toughness increase, in terms of areas related to the stress-displacement curve for ligament stretching.

is given by the shaded area between the stress-displacement relation for the bridging ligaments and the constant stress line representing the applied stress. The critical condition for matrix cracking is determined by the applied stress for which this area is equal to $J_c/2$ (i.e. $K_c^2(1-\nu^2)/2E$). Thus, for a given matrix and volume fraction of reinforcement this area is constant. It also follows that any change in the matrix toughness or composite stiffness that increases the ratio K_c/E must cause σ_c to increase. Moreover, the effect of changing the nature of the bridging ligaments on the matrix cracking stress can be readily deduced; generally, any change that stiffens the loading portion of the curve $p(u)$ must increase σ_c, whereas changes to the maximum value of $p(u)$ or to the region of the curve beyond the peak have no influence on σ_c. It is also immediately evident that, if the bridging mechanism is such that a residual stress simply translates the increasing portion of the curve $p(u)$ along the stress axis by σ_R^0 without changing its shape, the matrix cracking stress must increase by σ_R^0.

It is useful to contrast this representation for the steady-state matrix cracking stress with an analogous interpretation for the steady state *toughness* that results when σ_c exceeds the peak value of $p(u)$ and fibers break in the wake of the crack. In this case an expression for the steady state toughness increment ΔJ_c due to the bridging zone that remains over a limited region behind the crack tip has also been derived using the J-integral (Budiansky et al., 1986; Rose, 1987):

$$\Delta J_c = 2\int_0^{u_d} p(u)\,\mathrm{d}u \tag{29}$$

where u_d is the crack opening displacement above which the ligaments no longer restrain the crack. Therefore, the increase in fracture toughness is represented by the area *beneath* the curve $p(u)$ as depicted in Fig. 4(b). Any change that increases this area, including modification of the peak or the unloading portions of the curve $p(u)$, leads to an increase in fracture toughness. In contrast to the result for the steady-state matrix cracking stress, a *reduction* in stiffness of the loading portion of the curve $p(u)$ would usually lead to an increased toughness increment (provided the peak value of $p(u)$, i.e. the ligament strength, is not decreased). The influence of residual stress on the toughening increment is very dependent on the nature of the bridging ligaments because residual stresses would generally be expected to affect the peak of the curve $p(u)$ as well as translating the curve along the stress axis (Marshall and Evans, 1988). Thus, residual stresses of a given sign can either increase or decrease the degree of toughening depending on the bridging mechanisms.

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